Noninvertible Solitonic Symmetry

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Based on arXiv:2210.13780 with Shi Chen (U. Tokyo)

Conservation law of topological solitons is NOT fully characterized by homotopies $\pi_*(\mathcal{M})$.
Its algebraic structure can be more complicated $\Longrightarrow$ Noninvertible Solitonic Symmetry

Outline

1. Introduction

2. 4d $\mathbb{C}P^1$ σ-model & Hopfion

3. Potential application to $SO(N_c)$ QCD (ongoing)

4. Summary

When (usual) symmetry is spontaneously broken

$$G \xrightarrow{SSB} H,$$

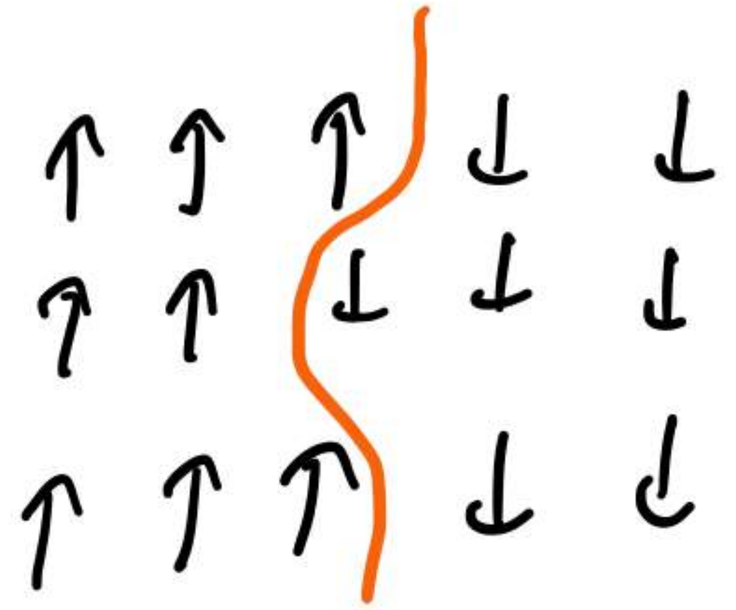
the low-energy effective theory is given by a non-linear σ-model:

$$\sigma : \text{Spacetime} \longrightarrow G/H$$

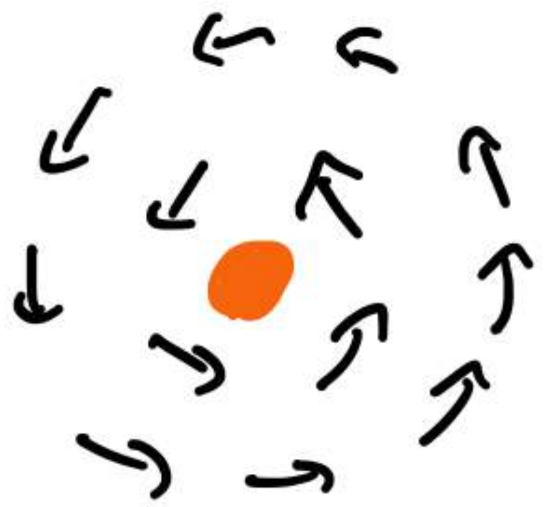
$$Z = \int \mathcal{D}\sigma \, \exp\left(-\frac{1}{g^2} \int |d\sigma|^2 + (\cdots) \right).$$

Small fluctuations of σ : Nambu-Goldstone modes

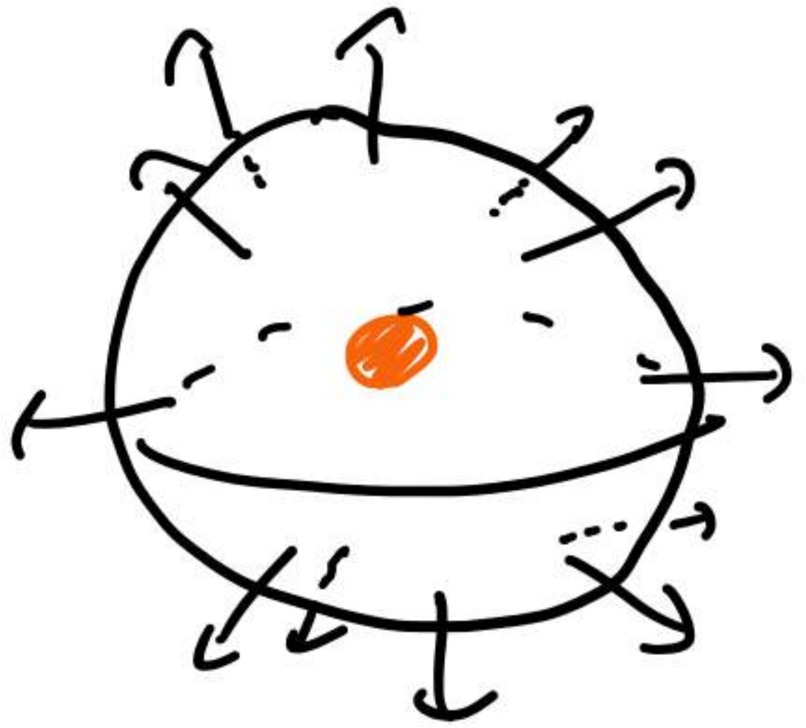
Large fluctuation : Topological Solitons



domain wall



vortex

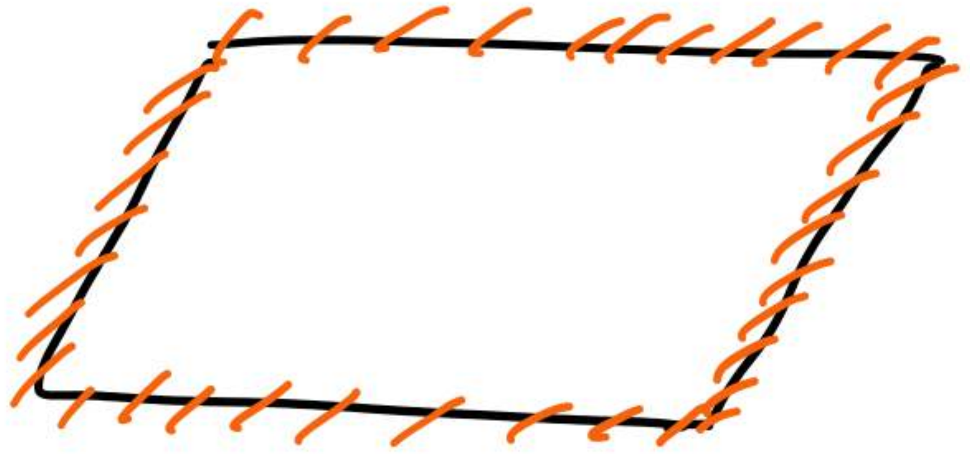


hedgehog

Topological Stability and Homotopies

(Mermin '79 Rev. Mod. Phys.)

Spacetime



$$\xrightarrow{\sigma} \mathcal{M} \; (= G/H).$$

To find finite action/energy (density) $\int |d\sigma|^2 < \infty$, its convenient to identifiy ∞'s of $\mathbb{R}^n$:

$$\mathbb{R}^n \cup \{\infty\} \simeq S^n \xrightarrow{\sigma} \mathcal{M}.$$

$\Rightarrow$ Topoligcal solitons are classified by homotopies of the target space

$$\pi_n(\mathcal{M}).$$

Recall that

$$\text{Conservation Law} \Longleftrightarrow \text{Symmetry},$$

we should be able to understand this topological conservation law as the symmetry of the σ-model.

Is this always true?

$$\boxed{\text{Conventional Wisdom: Solitonic Sym.} \cong \mathrm{Hom}(\pi_n(\mathcal{M}), U(1)).}$$

4d $\mathbb{C}P^1$ σ-model

Assume some (3+1) d quantum systems have SSB

$$SU(2) \xrightarrow{SSB} U(1).$$

The target space of the nonlinear σ-model becomes

$$\mathbb{C}P^1 \simeq SU(2)/U(1).$$

Lagrangian:

$$\begin{cases} \vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} & : \mathbb{C}^2\text{-valued scalar field with } |\vec{z}|^2 = 1. \\ a = a_\mu dx^\mu & : \text{(auxiliary) } U(1) \text{ gauge field} \end{cases}$$

$$\mathcal{L} = \frac{1}{g^2} \left| (\partial_\mu + i a_\mu) \vec{z} \right|^2.$$

This $U(1)$ gauge field a is auxiliary because its EoM can be solved as

$$a = i \vec{z}^\dagger \cdot d\vec{z}$$

Homotopy of $\mathbb{C}P^1$ $(\simeq S^2)$:

$$\pi_1(\mathbb{C}P^1) \simeq 0, \quad \underline{\pi_2(\mathbb{C}P^1) \simeq \mathbb{Z}}, \quad \underline{\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}}$$

"magnetic skyrmion"
"monopole"

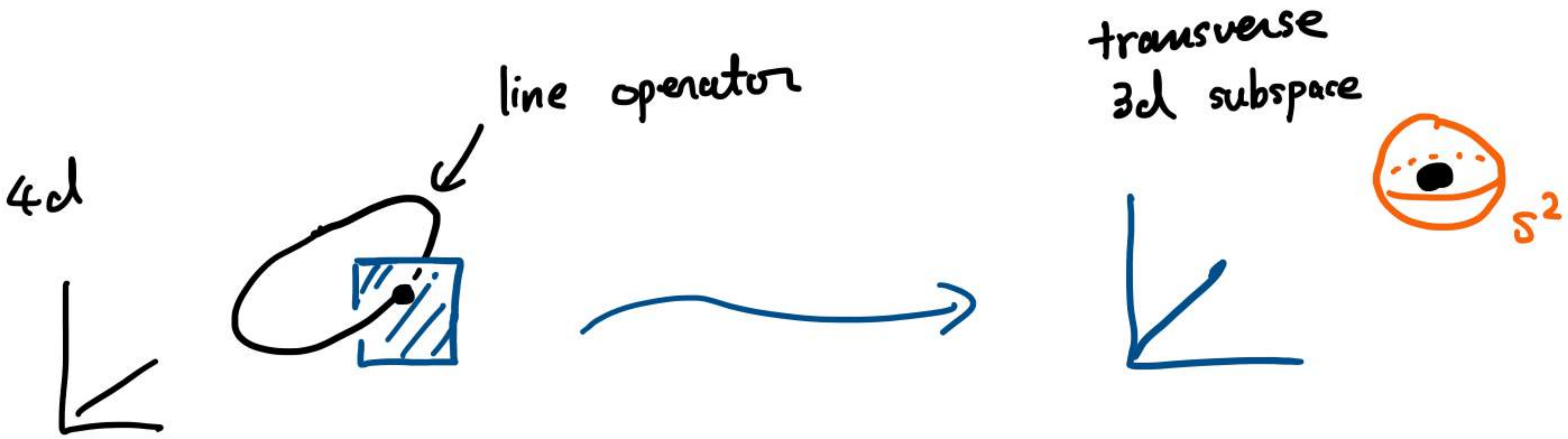
"Hopfion"

Vortex Soliton $\pi_2(\mathbb{CP}^1)$

$\pi_2(\mathbb{CP}^1) \simeq \mathbb{Z}$ has the Noether current

$$j = \frac{1}{2\pi} da,$$

and it gives $U(1)$ 1-form symmetry.



$$\underbrace{\int_{S^2} \frac{da}{2\pi} \in \mathbb{Z}}_{\text{Dirac quantization}} \simeq \pi_2(\mathbb{CP}^1)$$

We denote this line operator as $V_n(L)$ with $n = \int_{S^2} \frac{da}{2\pi}$.

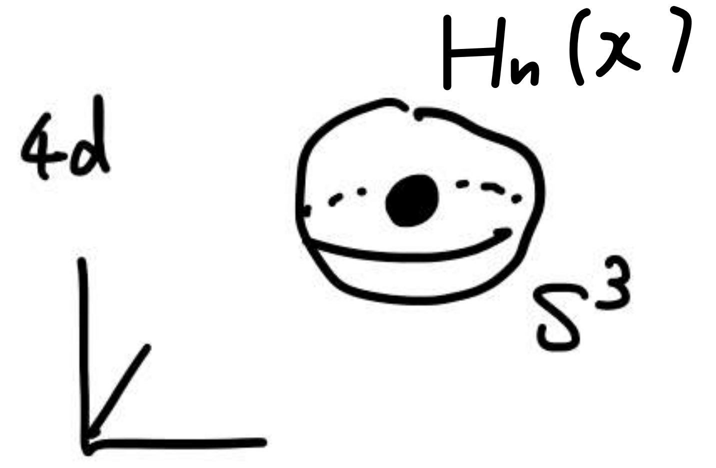
(We'll see that there is a finer classification for the vortex operators $V_n(L)$)

Hopfion $\pi_3(\mathbb{C}P^1)$

$\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}$ follows from the Hopf fibration $S^1 \to S^3 \to S^2$, and the corresponding solitons are known as "Hopfion" (or "Hopf soliton").

$\pi_3(\mathbb{C}P^1)$ is measured by the Hopf number:

$$\frac{1}{4\pi}\int_{S^3} a\,da \in \pi\,\mathbb{Z}$$



(* For general $U(1)$ gauge fields, the Chern-Simons form $\int \frac{1}{4\pi} a\,da$ can take arbitrary numbers.
Here, since a is an "auxiliary" field $(a = i\vec{z}^\dagger d\vec{z})$,
$d\left(\frac{1}{4\pi} a\,da\right) = \frac{1}{4\pi}(da)^2 = 0$. $\Rightarrow \int \frac{1}{4\pi} a\,da$ becomes quantized.)

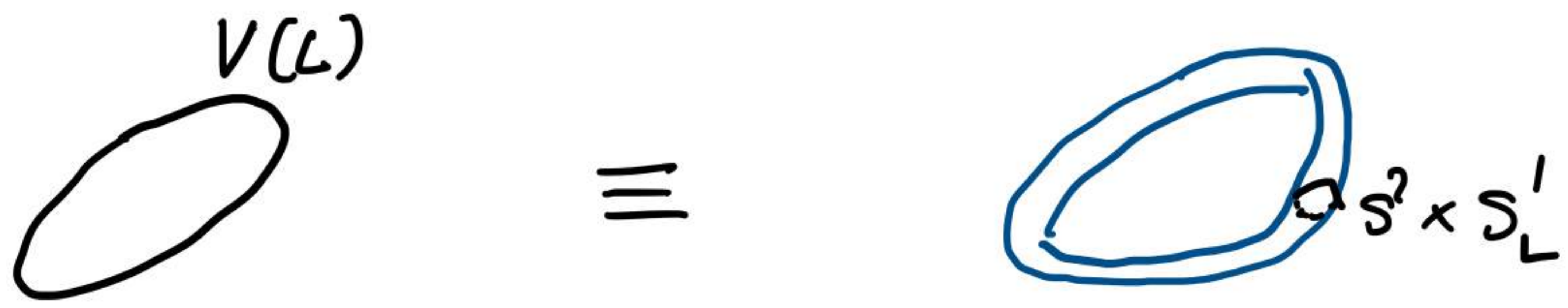
Unlike the case of $\pi_2(\mathbb{C}P^1)$, however, the integrand

$$\frac{1}{4\pi}\, a\,da$$

is not gauge invariant.

It's bizarre: Do we have $U(1)$ without Noether current?

Bizarre property of Hopfion charge : More on vortex operators $V_{n,k}(L)$.



Vortex operator at L
$\equiv$ Imposing the boundary condition for the torus $S^2 \times S^1_L$ around L.

Let's classify the $\mathbb{CP}^1$ configurations

$$S^2 \times S^1 \xrightarrow{\sigma} \mathbb{CP}^1$$

up to homotopy for a given monopole charge $\int_{S^2} \frac{da}{2\pi} = n$.

This can be classified by evaluating the Hopfion number on $S^2 \times S^1$:

$$k = \int_{S^2 \times S^1} \frac{ada}{4\pi^2}.$$

However, let's perform the large $U(1)$ gauge transformation $a \mapsto a + \varepsilon^{(1)}$ along S^1,

$$\int_{S^2\times S^1} \frac{ada}{4\pi^2} \mapsto \int_{S^2 \times S^1} \frac{ada}{4\pi^2} + \frac{1}{\pi} \underbrace{\int_{S^1} \varepsilon^{(1)}}_{\in 2\pi\mathbb{Z}} \underbrace{\int_{S^2} \frac{da}{2\pi}}_{=n} = \int_{S^2\times S^1} \frac{ada}{4\pi^2} + 2n\mathbb{Z}.$$

$\Rightarrow \int_{S^2 \times S^1} \frac{ada}{4\pi^2} = k$ is well-defined only in $\mathbb{Z}_{2n}$, i.e. $k \sim k + 2n$.

(We denote $V_{n,k}(L)$ for the vortex operator).

[cf. Pontrjagin '41]

Is the Hopfion symmetry $U(1)$ or $\mathbb{Z}_2$?

Let's evaluate correlation functions in a compact spacetime.

$$\left\langle H_{k_1}(x_1) \; H_{k_2}(x_2) \; H_{k_3}(x_3) \right\rangle \neq 0 \implies k_1 + k_2 + k_3 + \cdots = 0 .$$

$U(1)$ conservation law

$$\left\langle H_{k_1}(x_1) \; H_{k_2}(x_2) \; V_{n,k}(L) \right\rangle \neq 0 \implies k_1 + k_2 + \cdots + k = 0 \mod 2n .$$

$\mathbb{Z}_{2n}$ conservation law.

- Without $V(L)$, the Hopfion number conserves as if there is a $U(1)$ symmetry.
- With $V(L)$, the conservation law reduces to that of $\mathbb{Z}_2$.

Which is the symmetry group? Or, is it something else?

Generalized Symmetry in QFTs

$\boxed{\text{Generalized Symmetry} \quad = \quad \text{Topological Operators}}$

For continuous symmetry,

$Q(M_{d-1}) = \int_{M_{d-1}} \mathcal{L}$ is invariant under any topological continuous deformation of M_{d-1}.

In conventional symmetry, those topological operators obey group structures.

However, this turns out to be too restrictive to explore QFTs.

Non-invertible symmetry (or Categorical symmetry)

$$a \quad b \quad = \sum_c N^c_{ab}(M_{d-1}) \quad c$$

Fusion rule of symmetry defects can be quite general.

(2d CFTs : Verlinde '88, Bhadwaj, Tachikawa '17, Thorngren, Wang '19, ...

Higher dims (Since '21) : Nguyen, YT, Ünsal ; Heidenreich, McNamara, Reece, Rudelius, Valenzuela; Koide, Nagoya, Yamaguchi ; Choi, Cordova, Hsin, Lam, Shao ; Kaidi, Ohmori, Zheng ; ...)

Topological operators and TQFTs

How to find such topological operators for "unconventional" symmetries?

One of useful methods: (cf. Choi, Lam, Shao '22; Cordova, Ohmori '22)

1. Prepare a TQFT
2. Put it on a submanifold with a topological coupling to dynamical fields.

(As every ingredient is topological, this operator is manifestly topological. We should check if it acts nontrivially to local operators.)

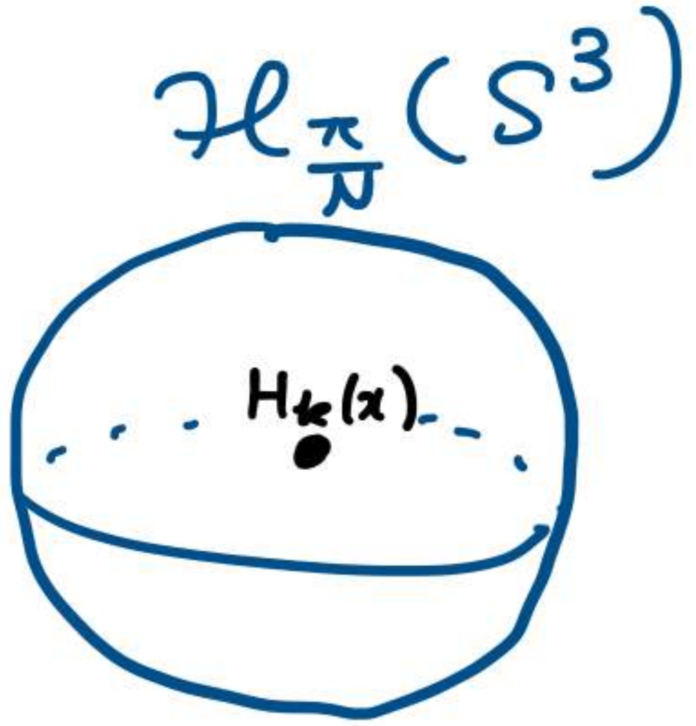
In our case,

1. We prepare the level-N $U(1)$ CS theory $\int \mathcal{D}b\, e^{i\frac{N}{4\pi}\int b db}$
2. The Hopfion symmetry operator is then defined as

$$\mathcal{H}_{\frac{\pi}{N}}(M_3, da) = \int \mathcal{D}b \exp\left(i\frac{N}{4\pi}\int_{M_3} b\wedge db + i\frac{1}{2\pi}\int_{M_3} b\wedge da \right).$$

Let's check how this operator acts on $\underbrace{H_k(x)}_{\text{Hopfion op}}$ and $\underbrace{V_{n,k}(L)}_{\text{vortex op.}}$.

Action of $\mathcal{H}_{\frac{\pi}{N}}(M_3)$ on $H_k(x)$



To evaluate the action of $\mathcal{H}_{\frac{\pi}{N}}(M_3)$, we can set $M_3 = S^3$ that surrounds x.

$$\mathcal{H}_{\frac{\pi}{N}}(S^3) = \int \mathcal{D}b \, e^{i\frac{N}{4\pi}\int b\,db + i\frac{1}{2\pi}\int b\,da}$$

$$= \int \mathcal{D}b \, e^{i\frac{N}{4\pi}\int_{S^3}(b+\frac{a}{N})\,d(b+\frac{a}{N})} \cdot e^{-i\frac{\pi}{N}\int_{S^3}\frac{a\,da}{4\pi^2}}$$

On S^3, $U(1)$ bundle is trivial.

$$\propto e^{-i\frac{\pi}{N}\int_{S^3}\frac{a\,da}{4\pi^2}}.$$

This shows that

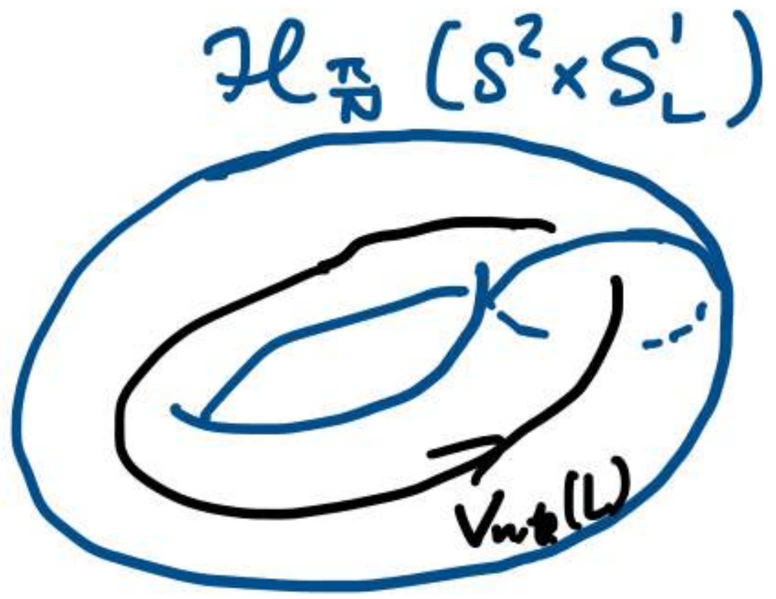
$$\langle \mathcal{H}_{\frac{\pi}{N}}(S^3)\, H_k(x) \cdots \rangle \propto e^{-i\frac{\pi}{N}k} \langle H_k(x) \cdots \rangle.$$

and $\mathcal{H}_{\frac{\pi}{N}}(S^3)$ detects the Hopfion charge of $H_k(x)$.

Since N can be arbitrary, $\{\mathcal{H}_{\frac{\pi}{N}}(S^3)\}_{N\geq 1}$ determines k as an integer.

$\Rightarrow$ Recovery of $U(1)$-like selection rule

Action of $\mathcal{H}_{\frac{\pi}{N}}(M_3)$ on $V_{n,k}(L)$



Next, we set $M_3 = S^2\times S^1_L$ to evaluate its action on $V_{n,k}(L)$.

$$\mathcal{H}_{\frac{\pi}{N}}(S^2\times S^1) = \int \mathcal{D}b\, e^{i\frac{N}{4\pi}\int b\,db + i\frac{1}{2\pi}\int b\,da}$$

U(1) bundle over $S^2\times S^1$ can be nontrivial.

$$= \sum_m (\cdots)\ \underbrace{\int_0^{2\pi} d\beta\, e^{i\overset{\int_{S^1} b}{\beta}(N\overset{\int_{S^2}\frac{db}{2\pi}}{m} + \overset{\int_{S^2}\frac{da}{2\pi}}{n})}}_{=0 \text{ unless } Nm+n=0.}$$

$$= \begin{cases} e^{-i\frac{\pi}{n}k} & \text{for } N=n. \\ 0 & \text{if } N \text{ is not a divisor of } n. \end{cases}$$

When the symmetry looks to be reduced to $\mathbb{Z}_{2n}$ by the presence of vortex operators, $\mathcal{H}_{\frac{\pi}{N}}(S^2\times S^1)$ acts nontrivially only if it fits the periodicity of $\mathbb{Z}_{2n}$.

Moreover, $\mathcal{H}_{\frac{\pi}{n}}(S^2\times S^1)$ captures the Hopfion charge of $V_{n,k}(L)$ in mod $2n$.

For 4d $\mathbb{C}P^1$ σ-model,

the Hopfion symmetry associated with $\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}$ is neither $U(1)$ nor $\mathbb{Z}_2$.

The correct symmetry generator is given by

$$\mathcal{H}_{\frac{\pi}{N}}(M_3) = \int \mathcal{D}b \, e^{i\frac{N}{4\pi}\int_{M_3} b\wedge db + i\frac{1}{2\pi}\int_{M_3} b\wedge da},$$

and the fusion rule is controlled by those of 3d TQFTs.

There is an invertible $\mathbb{Z}_2$ subgroup, generated by

$$\mathcal{H}_{\pi}(M_3) = e^{i\pi\int_{M_3}\frac{a\,da}{4\pi^2}} \in \mathbb{Z}_2 \left(\simeq \tilde{\Omega}_3^{\mathrm{Spin}}(\mathbb{C}P^1)\right).$$

It seems that the solitonic symmetry becomes non-invertible if there is a mismatch between homotopies and bordisms.

Potential application to $SO(N_c)$ QCD (Speculative)

Consider 4d $SO(N_c)$ gauge theory with N_f-flavor quarks in vector rep.

When N_c is large enough, we expect SSB of chiral symmetry:

$$SU(N_f) \xrightarrow{SSB} SO(N_f)$$

The low-energy theory is the $\frac{SU(N_f)}{SO(N_f)}$ non-linear σ-model.

QCD has baryons, whose mass is

$$M \sim N_c \Lambda.$$

Such a heavy object is usually understood as Skyrmions, classified by

$$\pi_3\left(\frac{SU(N_f)}{SO(N_f)}\right) \cong \begin{cases} \mathbb{Z} & (N_f = 2) \\ \mathbb{Z}_4 & (N_f = 3) \\ \mathbb{Z}_2 & (N_f \geq 4). \end{cases}$$

However, $SO(N_c)$ QCD always has $\mathbb{Z}_2$ baryon number symmetry. [Witten '83]

The UV and IR description has the discrepancy for $N_f = 2, 3$.

To my best knowledge,

this discrepancy is not yet resolved from the viewpoint of effective field theories.

(* In holographic setup, there is an insightful work by Imoto, Sakai, Sugimoto, suggesting the importance of pair creation/annihilation of D-branes.)

Similar?

Our result of 4d CP^1 σ-model suggests

the Hopfion # conservation can be explicitly broken by vortex creation/annihilation.

It may propose a new scenario for matching the UV/IR descriptions of 4d gauge theories.

Summary

- Topological conservation law for solitons is not fully characterized by Homotopies.

- 4d $\mathbb{C}P^1$ σ-model is carefully examined.

$$\begin{cases} \pi_2(\mathbb{C}P^1) \simeq \mathbb{Z} & \Rightarrow \ U(1) \ \text{1-form symmetry for vortex.} \\ \pi_3(\mathbb{C}P^1) \simeq \mathbb{Z} & \cancel{\Rightarrow} \ U(1) \ \text{symmetry for Hopfion.} \end{cases}$$

$$\left\langle H_{k_1}(x_1) \quad H_{k_2}(x_2) \quad V_{n,k}(L) \right\rangle \neq 0 \ \Rightarrow \ k_1 + k_2 + \cdots + k = 0 \mod 2n.$$

- The symmetry generators are given by 3d TQFT partition functions

$$\mathcal{H}_{\frac{\pi}{N}}(M_3) = \int \mathcal{D}b \ \exp\left(i \int_{M_3} \frac{N}{4\pi} b\, db + i \int_{M_3} \frac{1}{2\pi} b\, da\right).$$

$\Rightarrow$ Noninvertible Solitonic Symmetry

- It may have an interesting application to the dynamics of 4d $SO(N_c)$ QCD.
(Speculative)